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Asgeirsson's Mean Value Theorem and Related Identities

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ferential equation $(\Delta_x - \Delta_y)\, u = 0$ in a neighborhood of the convex compact set $K_R = \{(x, y);\, x, y \in \mathbf{R}^\nu,\, |x| + |y| \leqslant R\}$, then $\langle u, f \rangle = 0$ where f is the difference between the surface measures on the spheres $\{(x, 0);\, x \in \mathbf{R}^\nu,\, |x| = R\}$ and $\{(0, y);\, y \in \mathbf{R}^\nu,\, |y| = R\}$. We extend this to solutions of the inhomogeneous equation by proving that $f = (\Delta_x - \Delta_y)\, \mu_R$ where $\mu_R = \frac{1}{2} \chi_+^{(1-\nu)/2} (\pi A_R / 4R^2)$, with $A_R(x, y) = (|x|^2 - |y|^2)^2 - 2R^2(|x|^2 + |y|^2) + R^4$. The distributions χ_+^a on $\mathbf{R}$ are defined by $\chi_+^a(t) = t_+^a / \Gamma(a+1)$ when $\operatorname{Re} a > -1$ and then continued analytically to all $a \in \mathbf{C}$. This formula is closely related to the fundamental solution of the wave equation in $\mathbf{R}^{\nu+1}$. Similar identities are given for arbitrary indefinite nonsingular real quadratic forms in $\mathbf{R}^{2\nu}$. This work was originally motivated by the progressive solutions of the wave equation given by G. Friedlander and M. Riesz.

1. INTRODUCTION

There are many proofs of Asgeirsson's mean value theorem for solutions of the ultrahyperbolic equation. In Section 2 we shall give one which makes explicit the correction term required when the equation is inhomogeneous. The proof shows that the mean value theorem in $\mathbf{R}^{2\nu}$ is closely related to the wave equation in $\mathbf{R}^{\nu+1}$. A general form of this observation is given in Section 3. As an application we prove in Section 4 results similar to Asgeirsson's theorem for an arbitrary indefinite nonsingular metric in $\mathbf{R}^{2\nu}$. For $\mathbf{R}^4$ with the Lorentz metric they are closely related to the progressive solutions of the wave equation in Friedlander [F] in the form given to them by Riesz [R].

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2. ASGEIRSSON'S THEOREM

The theorem states that if u is a continuous solution of the ultrahyperbolic equation $(\Delta_x - \Delta_y)\, u = 0$ in a neighborhood of the convex compact set $K_R = \{(x, y); x, y \in \mathbf{R}^\nu, |x| + |y| \leqslant R\} \subset \mathbf{R}^{2\nu}$ then

$$\int_{|x|=R} u(x, 0)\, dS(x) - \int_{|y|=R} u(0, y)\, dS(y) = 0. \tag{2.1}$$

Here dS is the surface measure on the Euclidean sphere of radius R in $\mathbf{R}^\nu$. The proof given in [H1, p. 184] was based on the observation that it follows easily by Fourier analysis that the distribution f_R defined by

$$\langle f_R, \varphi \rangle = \int_{|x|=R} \varphi(x, 0)\, dS(x) - \int_{|y|=R} \varphi(0, y)\, dS(y), \qquad \varphi \in C_0^\infty(\mathbf{R}^{2\nu}), \tag{2.2}$$

is equal to $(\Delta_x - \Delta_y)\, \mu_R$ for some $\mu_R \in \mathscr{E}'(K_R)$, which implies (2.1). Since $\mu_R = E * f_R$ where E is a fundamental solution of $\Delta_x - \Delta_y$, it was also proved using well known fundamental solutions (cf. Proposition 3.1) that $\operatorname{supp} \mu_R \subset \partial K_R$ when ν is odd, which implies that (2.1) is then valid for all continuous u satisfying the ultrahyperbolic equation just in a neighborhood of ∂K_R. When ν is even it was proved that $\operatorname{sing\,supp}_A \mu_R \subset \partial K_R$ and that $\mu_R(0, 0) \neq 0$; hence $\operatorname{supp} \mu_R = K_R$. For $\nu = 3$ an earlier proof by H. Lewy [L] shows that μ_R is then a simple layer on ∂K_R. Lewy mentions that Asgeirsson has stated that he had obtained the same result and an extension to higher dimensions. This has not been published but S. Helgason has informed me that there was an outline in a letter to him from Asgeirsson in 1955. We shall here determine μ_R explicitly for arbitrary ν.

The interior of K_R is the component of the origin in the set where the polynomial

$$A_R(x, y) = (|x|^2 - |y|^2)^2 - 2R^2(|x|^2 + |y|^2) + R^4 \tag{2.3}$$

is not equal to 0, for we have

$$A_R(x, y) = (R - |x| - |y|)(R - |x| + |y|)(R + |x| - |y|)(R + |x| + |y|). \tag{2.4}$$

Since

$$\partial A_R(x, y)/\partial x = 4(|x|^2 - |y|^2 - R^2)\, x, \quad \partial A_R(x, y)/\partial y = 4(|y|^2 - |x|^2 - R^2)\, y,$$

the two spheres $S_1=\{(x,0); |x|=R\}$ and $S_2=\{(0,y); |y|=R\}$ in (2.2) are the only singular points of $\Sigma_R=\{(x,y); A_R(x,y)=0\}$, and since

$$|\partial A_R(x,y)/\partial x|^2-|\partial A_R(x,y)/\partial y|^2=16(|x|^2-|y|^2)\,A_R(x,y),$$

$$\Delta_x A_R(x,y)-\Delta_y A_R(x,y)=8(|x|^2-|y|^2)(\nu+1),$$

the regular part of Σ_R is a characteristic surface, the bicharacteristics are the lines joining points on S_1 and S_2, and

$$(\Delta_x-\Delta_y)\,f(A_R)=8(|x|^2-|y|^2)(f'(A_R)(\nu+1)+2A_R f''(A_R))$$

vanishes if f' is homogeneous of degree $-(\nu+1)/2$, hence if f is homogeneous of degree $(1-\nu)/2$. We shall prove that μ_R is a multiple of $f(A_R)$ in K_R if $f=\chi_+^{(1-\nu)/2}$ and the composition is suitably defined at the two singular spheres on Σ_R where the preceding calculation is not valid or even defined. Here the distribution χ_+^a on $\mathbf{R}$ is defined as the function $t\mapsto t_+^a/\Gamma(a+1)$ if $a\in\mathbf{C}$ and $\operatorname{Re} a>-1$ and is extended analytically to all $a\in\mathbf{C}$ by the equation $(\chi_+^{a+1})'=\chi_+^a$.

To examine A_R closely at the sphere $S_1=\{(x,0); |x|=R\}$, we introduce polar coordinates with respect to x, writing $x=(R+s)\,\alpha$ where $\alpha\in S^{\nu-1}=\{\alpha\in\mathbf{R}^\nu; |\alpha|=1\}$ and $s>-R$. Then

$$A_R(x,y)/4R^2=((R-|x|)^2-|y|^2)((R+|x|)^2-|y|^2)/4R^2$$
$$=(s^2-|y|^2)\,G(s,y),$$

where $G(s,y)=((2R+s)^2-|y|^2)/4R^2$ is equal to 1 at the origin. Thus a neighborhood of S_1 becomes with these coordinates the product of $S^{\nu-1}$ and a neighborhood of the origin in $\mathbf{R}^{\nu+1}$, while $A_R/4R^2$ is a positive multiple of the Lorentz form in $\mathbf{R}^{\nu+1}$. The forward fundamental solution F of $\partial_s^2-\Delta_y$ is equal to $\frac{1}{2}\chi_+^{(1-\nu)/2}(\pi(s^2-|y|^2))$ when $s>0$, equal to 0 when $s<|y|$, and is extended as a homogeneous distribution of degree $(1-\nu)$ in $\mathbf{R}^{1+\nu}$ (cf. [H1, p. 140]). The homogeneity implies that $\psi(s/\varepsilon)\,F$ and $\psi(s/\varepsilon)\,F'\to 0$ as $\varepsilon\to 0$ if $\psi\in C_0^\infty(\mathbf{R})$. Thus $G(s,y)^{(1-\nu)/2}\,F$ is an extension of $\frac{1}{2}\chi_+^{(1-\nu)/2}(\pi A_R/4R^2)$ to a neighborhood of S_1, with support in K_R, and we define similarly an extension to a neighborhood of $S_2=\{(0,y); |y|=R\}$. We shall prove that the extension M thus defined is equal to μ_R. In $K_R\backslash(S_1\cup S_2)$ we know that it satisfies the homogeneous ultrahyperbolic equation. To apply the ultrahyperbolic operator at S_1 we write

$$\Delta_x-\Delta_y=\frac{\partial^2}{\partial s^2}+\frac{\nu-1}{R+s}\,\frac{\partial}{\partial s}+\frac{1}{(R+s)^2}\,\Delta_\alpha-\Delta_y,$$

where Δ_α is the Laplace operator in S^{n-1}, and obtain in the α, s, y coordinates in a neighborhood of S_1

$$(\Delta_x - \Delta_y) M - \delta(s)\,\delta(y) = G_0(s, y)\,\partial F/\partial s + \sum_1^\nu G_j(s, y)\,\partial F/\partial y_j + H(s, y)\,F,$$

where G_j and H are in C^∞. The right-hand side vanishes for $(s, y) \neq 0$ so the product by $(1-\psi(s/\varepsilon))$ is equal to 0 if $\psi \in C_0^\infty(\mathbf{R})$ and $\psi = 1$ in a neighborhood of 0. On the other hand, the product by $\psi(s/\varepsilon)$ converges to 0 when $\varepsilon \to 0$, which proves that it is equal to 0, so $M = \mu_R$. In the original variables $\delta(s)$ is the area measure on S_1, so we have proved most of the following extension of Asgeirsson's theorem:

THEOREM 2.1. *The distribution* $f_R \in \mathscr{E}'(\mathbf{R}^{2\nu})$ *in* (2.2) *is equal to* $(\Delta_x - \Delta_y)\,\mu_R$ *where* $\mu_R \in \mathscr{E}'(K_R)$ *is the limit as* $\varepsilon \to 0$ *of*

$$(1-\psi((|x|-R)/\varepsilon))(1-\psi((|y|-R)/\varepsilon))\ \tfrac{1}{2}\chi_+^{(1-\nu)/2}(\pi A_R/4R^2),$$

restricted to $\mathscr{E}'(K_R)$, *if* $\psi \in C_0^\infty(\mathbf{R})$ *and* $\psi = 1$ *in* $(-1, 1)$. *This is a distribution of order* $[(\nu-2)/2]$, *and we have*

$$\int_{|x|=R} u(x, 0)\, dS(x) - \int_{|y|=R} u(0, y)\, dS(y) = \langle \mu_R, \Delta_x u - \Delta_y u \rangle \tag{2.5}$$

if u *is continuous in a neighborhood of the spheres in the left-hand side and* $\Delta_x u - \Delta_y u \in C^{[(\nu-2)/2]}$ *in a neighborhood of* supp μ_R, *that is*, ∂K_R *if* ν *is odd and* K_R *if* ν *is even.*

Proof. We have already proved (2.5) when $u \in C^\infty$, and the formula follows under the weaker assumption by applying it to regularizations of u.

3. FUNDAMENTAL SOLUTIONS AND CRITICAL MANIFOLDS

The main point in the proof of Theorem 2.1 above was that the polynomial A_R defined by (2.3) is a positive multiple of the Lorentz form transversally to the spheres which constitute the critical set of A_R. In this section we shall generalize this setup, but first we have to recall some fundamental solutions of more general second order differential operators (see e.g. [H1, Theorem 6.2.1]).

PROPOSITION 3.1. *Let* $\mathscr{A}$ *be a nonsingular real quadratic form in* $\mathbf{R}^n$ *with signature* (n_+, n_-). *If* $c_n = 2\pi^{n/2}/\Gamma(n/2)$ *is the area of the unit sphere in* $\mathbf{R}^n$ *and* $n > 2$ *then*

$$\mathscr{A}^*(\partial)(\mathscr{A} \pm i0)^{(2-n)/2} = (2-n)\, c_n\, |\det A|^{-1/2}\, e^{\mp \pi i n_-/2} \delta_0, \tag{3.1}$$

$$\mathscr{A}^*(\partial)\, \chi_\pm^{(2-n)/2}(\mathscr{A}) = \pm 4\pi^{(n-2)/2} \sin(\pi n_\pm/2)\, |\det A|^{-1/2}\, \delta_0. \tag{3.2}$$

Here A is the symmetric matrix of $\mathscr{A}$ and $\mathscr{A}^*$ is the dual quadratic form of $\mathscr{A}$ defined by A^{-1}. The distributions $(\mathscr{A} \pm i0)^{(2-n)/2}$ and $\chi_\pm^{(2-n)/2}(\mathscr{A})$ are the homogeneous distributions defined in $\mathbf{R}^n \setminus \{0\}$ by pullback of the distributions $(t \pm i0)^{(2-n)/2}$ and $\chi_\pm^{(2-n)/2}$ on $\mathbf{R}$ by the map $x \mapsto \mathscr{A}(x)$. Recall that $(t \pm i0)^a$ is defined as distributional boundary values of z^a defined for $z \in \mathbf{C} \setminus \mathbf{R}$ by $|\arg z| < \pi$, and that $\chi_\pm^a$ is the function defined by $t \mapsto t_\pm^a / \Gamma(a+1)$ when $\operatorname{Re} a > -1$ and is extended analytically to arbitrary $a \in \mathbf{C}$ by the equation $(\chi_\pm^{a+1})' = \pm \chi_\pm^a$. Since $(t \pm i0)^a$ is a basis for homogeneous distributions of degree a on $\mathbf{R}$ when a is not an integer $\geqslant 0$, it is clear that (3.2) must follow from (3.1), as was verified in [H1, p. 138].

It is convenient to make some remarks on Proposition 3.1 before stating a generalization.

(i) If $\psi_\varepsilon \in C_0^\infty(\mathbf{R}^n)$ and $\psi_\varepsilon(\varepsilon\,\cdot)$ is bounded in $C_0^\infty(\mathbf{R}^n)$, then $\psi_\varepsilon F \to 0$ in $\mathscr{D}'(\mathbf{R}^n)$ when $\varepsilon \to 0$ if $F \in \mathscr{D}'(\mathbf{R}^n)$ is homogeneous of degree $\mu > -n$, for if $\varphi \in C_0^\infty(\mathbf{R}^n)$ then $\langle \psi_\varepsilon F, \varphi \rangle = \varepsilon^{\mu+n} \langle F, \psi_\varepsilon(\varepsilon\,\cdot)\, \varphi(\varepsilon\,\cdot) \rangle$, and $\psi_\varepsilon(\varepsilon\,\cdot)\, \varphi(\varepsilon\,\cdot)$ is bounded in $C_0^\infty(\mathbf{R}^n)$. Thus $F = \lim_{\varepsilon \to 0} (1 - \psi_\varepsilon) F$, and if $\psi_\varepsilon = 1$ in a neighborhood of the origin this defines the homogeneous distribution F in terms of its restriction to $\mathbf{R}^n \setminus \{0\}$.

(ii) If $\mathscr{Q}(x, \partial_x)$ is a second order differential operator with C^∞ coefficients such that $\mathscr{Q}(x, \partial_x)(\mathscr{A} \pm i0)^{(2-n)/2} = 0$ in $\mathbf{R}^n \setminus \{0\}$ then the second order terms $\langle q \partial_x, \partial_x \rangle$ in $\mathscr{Q}(0, \partial_x)$ are equal to $\gamma \mathscr{A}^*(\partial_x)$ for some γ. In fact, replacing x by εx we obtain after multiplication by ε^n that

$$\varepsilon^2 \mathscr{Q}(\varepsilon x, \varepsilon^{-1} \partial/\partial x)(\langle Ax, x \rangle \pm i0)^{(2-n)/2} = 0 \qquad \text{in } \mathbf{R}^n \setminus \{0\},$$

and when $\varepsilon \to 0$ it follows that $\langle q \partial_x, \partial_x \rangle (\langle Ax, x \rangle \pm i0)^{(2-n)/2} = 0$. Taking q symmetric this means explicitly that

$$(2-n)(\operatorname{Tr}(qA)(\langle Ax, x \rangle \pm i0)^{-n/2} - n \langle qAx, Ax \rangle (\langle Ax, x \rangle \pm i0)^{-(n+2)/2}) = 0$$

when $x \neq 0$. Thus $\langle Ax, x \rangle \operatorname{Tr}(qA) = n \langle AqAx, x \rangle$, which means that $AqA = \gamma A$ and $q = \gamma A^{-1}$ where

$$\gamma = \operatorname{Tr}(qA)/n = \langle q \partial_x, \partial_x \rangle\, \mathscr{A}/2n = \mathscr{Q}(x, \partial_x)\, \mathscr{A}(x)/2n \qquad \text{when} \quad x = 0. \tag{3.3}$$

(iii) $\delta_0/\sqrt{|\det A|}$ does not change if we make a linear change of coordinates in $\mathbf{R}^n$. The Dirac "function" δ_0 is really a density, and division by $\sqrt{|\det A|}$ makes it transform as a function.

The following proposition gives the geometrical part of our extension of Proposition 3.1.

PROPOSITION 3.2. *Let X be a C^∞ manifold of dimension n and let $F \in C^\infty(X)$ be a real valued function such that*

(i) *$S = \{x \in X; F(x) = 0, F'(x) = 0\}$ is a C^∞ submanifold of codimension $\nu > 2$.*

(ii) *The Hessian $F''(x)$ has rank ν for every $x \in S$.*

Then there is a unique positive distribution $\delta_{S,F}$ with support on S such that if ψ is a C^∞ map from an open subset U of X to $\mathbf{R}^\nu$ with $\psi(x) = 0$ and $\psi'(x)$ surjective when $x \in S \cap U$, then $\delta_{S,F} = \delta_0(\psi)/\sqrt{|\det A|}$ in U where δ_0 is the Dirac "function" in $\mathbf{R}^\nu$ and A is the symmetric $\nu \times \nu$ matrix defined by ${}^t\psi'(x)\, A\psi'(x) = \frac{1}{2}F''(x)$ when $x \in S \cap U$.

Note that F'' can be regarded as a quadratic form in the normal bundle of S, that is, the quotient $T(X)|_S/T(S)$, with dual equal to the conormal bundle of S. This justifies the existence of A.

Proof. We only have to verify that the condition on $\delta_{S,F}$ is independent of the choice of ψ. This follows from the fact that the condition only depends on $\psi'(x)$ and that replacing $\psi(x)$ by $T\psi(x)$ where T is an invertible $\nu \times \nu$ matrix will change A to ${}^tT^{-1}AT^{-1}$ with determinant $\det A/(\det T)^2$ while $\delta_0(T\psi) = \delta_0(\psi)/|\det T|$.

We are now ready for a generalized version of Proposition 3.1:

THEOREM 3.3. *If the hypotheses of Proposition 3.2 are fulfilled, then there are uniquely defined distributions $u_\pm$ in X such that for every $\varrho \in C^\infty(X)$ with $\varrho = 0$ in S, $\varrho > 0$ in $X \setminus S$, and ϱ'' of rank ν on S we have*

$$u_\pm = \lim_{\varepsilon \to 0} (1 - \psi(\varrho/\varepsilon^2))(F \pm i0)^{(2-\nu)/2}, \tag{3.4}$$

if $\psi \in C_0^\infty(\mathbf{R})$ is equal to 1 in a neighborhood of 0. We shall use the notation $(F \pm i0)^{(2-\nu)/2}$ also for the extension and define $\chi_\pm^{(2-\nu)/2}(F)$ similarly. If $\mathcal{Q}$ is a second order differential operator with C^∞ coefficients such that $\mathcal{Q}u_\pm = 0$ in $X \setminus S$, then the principal symbol of $\mathcal{Q}$ restricted to the conormal bundle of S is equal to γ times the dual of the quadratic form defined by $F''/2$ in the normal bundle of S. Here

$$\gamma = \mathcal{Q}F/2\nu \qquad \text{on } S. \tag{3.5}$$

The signature (ν_+, ν_-) of F'' regarded as a quadratic form in the normal bundle of S is constant on every component of S, and

$$\mathcal{Q}(F \pm i0)^{(2-\nu)/2} = (2-\nu)\, c_\nu\, e^{\mp \pi i \nu_-/2} \gamma \delta_{S,F}, \tag{3.6}$$

$$\mathcal{Q}\chi_\pm^{(2-\nu)/2}(F) = \pm 4\pi^{(\nu-2)/2} \sin(\pi\nu_\pm/2)\, \gamma\delta_{S,F}. \tag{3.7}$$

Proof. In a neighborhood U of any point in S we can by the Morse lemma introduce local coordinates, which we now denote by (x, y), $x \in \mathbf{R}^\nu$, $y \in \mathbf{R}^{n-\nu}$, such that S is defined by $x=0$ and $F(x, y) = \langle Ax, x\rangle$ is a quadratic form with $\det A = \pm 1$. With these coordinates $\delta_{S,F} = \delta(x)$.

The distributions $(\langle Ax, x\rangle \pm i0)^{(2-\nu)/2}$ in $\mathbf{R}^\nu \setminus \{0\}$ are homogeneous of degree $2-\nu$ so they have unique homogeneous extensions $U_\pm$. With ψ and ϱ as in the theorem we have $\psi(\varrho(\varepsilon x, y)/\varepsilon^2) \to \psi(\frac{1}{2}\langle \varrho''_{xx}(0, y)\, x, x\rangle)$ in C_0^∞, as a function of x when $\varepsilon \to 0$, so (3.4) is valid locally with $u_\pm = U_\pm$ by remark (i) after Proposition 3.1. Since (3.4) is independent of the choice of coordinates this proves the existence of $u_\pm$.

Writing $\mathcal{Q} = Q(x, y, \partial_x, \partial_y)$ we have by hypothesis when $x \neq 0$

$$Q(x, y, \partial_x, \partial_y)(\langle Ax, x\rangle \pm i0)^{(2-\nu)/2} = 0.$$

If we replace x by εx, multiply by ε^ν and let $\varepsilon \to 0$, it follows as in remark (ii) after Proposition 3.1 that the second order part of $Q(0, y, \partial/\partial x, 0)$ is equal to $\gamma(y)\langle A^{-1}\partial_x, \partial_x\rangle$ with γ given by (3.5) as in (3.3). Thus the coefficients of the terms in $Q(x, y, \partial_x, \partial_y) - \gamma(y)\langle A^{-1}\partial_x, \partial_x\rangle$ of order 2 with respect to x must vanish when $x=0$, which proves that

$$\begin{aligned}(Q(x, y, \partial_x, \partial_y) &- \gamma(y)\langle A^{-1}\partial_x, \partial_x\rangle)\, u_\pm \\ &= \sum_{|\alpha| \leqslant 1} b_\alpha(x, y)\, \partial_x^\alpha u_\pm + \sum_{|\alpha|=2} \sum_{j=1}^{\nu} b_{\alpha j}(x, y)\, x_j \partial_x^\alpha u_\pm,\end{aligned}$$

where b_α and $b_{\alpha j}$ are in C^∞. By hypothesis the right-hand side vanishes when $x \neq 0$, so the product by $1 - \psi(|x|^2/\varepsilon^2)$ is equal to 0. Since $\partial_x^\alpha u_\pm$ is homogeneous of degree $2 - |\alpha| - \nu > -\nu$ when $|\alpha| \leqslant 1$ and $x_j \partial_x^\alpha u_\pm$ is homogeneous of degree $1-\nu$ if $|\alpha| = 2$, it follows from remark (i) after Proposition 3.1 that the product by $\psi(|x|^2/\varepsilon^2)$ converges to 0 when $\varepsilon \to 0$. Hence

$$Q(x, y, \partial_x, \partial_y)\, u_\pm(x, y) = \gamma(y)\langle A^{-1}\partial_x, \partial_x\rangle\, u_\pm = \gamma(y)(2-\nu)\, c_\nu e^{\mp \pi i \nu_-/2} \delta(x)$$

by (3.1), which completes the proof.

Remark 3.4. If F'' has Lorentz signature, that is, $\nu_+ = 1$, then $\{x \in X;\, F(x) > 0\}$ has locally two components, and $\chi_+^{(2-\nu)/2}(F)$ is the sum of two terms each supported by the closure of one of them and satisfying

(3.4). For these terms we have (3.6), (3.7) with the right-hand side divided by 2. (See [H1, p. 140].) If $-F''$ has Lorentz signature, that is, $\nu_- = 1$, then $\chi_-^{(\nu-2)/2}(F)$ splits in the same way.

4. GENERAL INDEFINITE QUADRATIC METRICS

Asgeirsson's mean value theorem as stated in Theorem 2.1 is a special case of Theorem 3.3, and it has a natural generalization to an arbitrary indefinite nondegenerate real quadratic form in a finite dimensional vector space V (of even dimension). Choose a $\mathcal{Q}$ orthogonal decomposition $V = V_1 \oplus V_2$ and denote elements in V by (x, y) where $x \in V_1$ and $y \in V_2$. Thus $\mathcal{Q}(x, y) = \mathcal{Q}_1(x) - \mathcal{Q}_2(y)$ where $\mathcal{Q}_j$ is a nondegenerate quadratic form in V_j. Since $\mathcal{Q}$ is indefinite we can choose the decomposition so that $\mathcal{Q}_j$ is not negative definite, $j = 1, 2$. We define nondegenerate quadrics in V_j by

$$S_1 = \{(x, 0); \mathcal{Q}_1(x) = R^2\}, \qquad S_2 = \{(0, y); \mathcal{Q}_2(y) = R^2\}, \tag{4.1}$$

where $R > 0$, and we form the union of the lines joining them,

$$\{(\lambda\alpha, \mu\beta); \mathcal{Q}_1(\alpha) = R^2, \mathcal{Q}_2(\beta) = R^2, \lambda + \mu = 1\}. \tag{4.2}$$

When $x = \lambda\alpha$, $y = \mu\beta$ we have $\mathcal{Q}_1(x) = \lambda^2 R^2$, $\mathcal{Q}_2(y) = \mu^2 R^2$, and if $\lambda + \mu = 1$ then

$$\begin{aligned}\mathcal{Q}(x, y)^2 &= (\lambda^2 - \mu^2)^2 R^4 = (\lambda - \mu)^2 R^4 = (2(\lambda^2 + \mu^2) - (\lambda + \mu)^2) R^4 \\ &= 2R^2(\mathcal{Q}_1(x) + \mathcal{Q}_2(y)) - R^4,\end{aligned}$$

which means that

$$\begin{aligned}T_R(x, y) &= \mathcal{Q}(x, y)^2 - 2R^2(\mathcal{Q}_1(x) + \mathcal{Q}_2(y)) + R^4 \\ &= (\mathcal{Q}(x, y) - R^2)^2 - 4R^2\mathcal{Q}_2(y) = (\mathcal{Q}(x, y) + R^2)^2 - 4R^2\mathcal{Q}_1(x)\end{aligned} \tag{4.3}$$

must vanish on (4.2). Conversely, if $T_R(x, y) = 0$ then $\mathcal{Q}_1(x) \geqslant 0$ and $\mathcal{Q}_2(y) \geqslant 0$, and

$$\begin{gathered}\sqrt{\mathcal{Q}_1(x)} + \sqrt{\mathcal{Q}_2(y)} = R, \qquad \text{or} \qquad \sqrt{\mathcal{Q}_1(x)} = R + \sqrt{\mathcal{Q}_2(y)}, \\ \text{or} \qquad \sqrt{\mathcal{Q}_2(y)} = R + \sqrt{\mathcal{Q}_1(x)}.\end{gathered} \tag{4.4}$$

If $Q_1(x) > 0$ and $Q_2(y) > 0$ then (x, y) is in the set (4.2) with $0 < \lambda < 1$, $\lambda > 1$ and $\lambda < 0$, respectively. If, say, $\mathcal{Q}_2(y) = 0$, hence $\mathcal{Q}_1(x) = R^2$, then we can find y' close to y with $\mathcal{Q}_2(y') > 0$ and choose x' close to x so that the first or second condition (4.4) is fulfilled, so the set (4.2) is dense in the set of all zeros of T_R. The first and second cases in (4.4) overlap if $\mathcal{Q}_2(y) = 0$ and $\mathcal{Q}_1(x) = R^2$, which happens only in S_1 if $\mathcal{Q}_2$ is positive definite. Similarly the first and third cases overlap only in S_2 if $\mathcal{Q}_1$ is positive definite.

The polynomial T_R is irreducible unless $\dim V_1 = 1$ or $\dim V_2 = 1$, for $\mathcal{Q}$ is irreducible when $\dim V \geqslant 3$, and if

$$T_R(x, y) = (\mathcal{Q}(x, y) + \mathcal{L}_1(x, y) + C_1)(\mathcal{Q}(x, y) + \mathcal{L}_2(x, y) + C_2),$$

where $\mathcal{L}_j$ are linear forms and C_j constants, then identification of the third and first order terms gives $\mathcal{L}_1 + \mathcal{L}_2 = 0$ and $C_1 = C_2$. The remaining equations

$$2C_1(\mathcal{Q}_1(x) - \mathcal{Q}_2(y)) - \mathcal{L}_1(x, y)^2 = -2R^2(\mathcal{Q}_1(x) + \mathcal{Q}_2(y)), \qquad C_1^2 = R^4,$$

give $C_1 = \pm R^2$ and $4R^2\mathcal{Q}_1(x) = \mathcal{L}_1(x, y)^2$ or $4R^2\mathcal{Q}_2(y) = \mathcal{L}_1(x, y)^2$ for the two signs, hence $\dim V_1 = 1$ or $\dim V_2 = 1$. If $\dim V_1 = 1$ we can assume that $V_1 = \mathbf{R}$ and that $\mathcal{Q}_1(x) = x^2$. Then we have a factorization $T_R(x, y) = ((x + R)^2 - \mathcal{Q}_2(y))((x - R)^2 - \mathcal{Q}_2(y))$, and it follows that

$$\{(x, y);\, T_R(x, y) = 0\} = \{(x, y);\, \mathcal{Q}(x - R, y) = 0\}$$
$$\cup \{(x, y);\, \mathcal{Q}(x + R, y) = 0\}.$$

To avoid the trivial and exceptional situation which occurs when T_R is reducible *we assume from now on that* $\dim V_j \geqslant 2$ *for* $j = 1, 2$.

As in the case of A_R in Section 2, the zero set of T_R is a smooth manifold except at the singular quadrics (4.1), and the regular part Σ_R is characteristic with respect to the differential operator $\mathcal{Q}^*(\partial) = \mathcal{Q}_1^*(\partial_x) - \mathcal{Q}_2^*(\partial_y)$ corresponding to the dual quadratic form of $\mathcal{Q}$ in V'. Using coordinates we write $\mathcal{Q}_1(x) = \langle Q_1 x, x\rangle$ where Q_1 is a symmetric matrix; then $\mathcal{Q}_1^*(\partial_x) = \langle Q_1^{-1}\partial_x, \partial_x\rangle$ and similarly for $\mathcal{Q}_2$. We have

$$\partial_x T_R(x, y) = 4(\mathcal{Q}(x, y) - R^2)\, Q_1 x, \quad \partial_y T_R(x, y) = -4(\mathcal{Q}(x, y) + R^2)\, Q_2 y,$$

which gives

$$\begin{aligned}\langle Q_1^{-1}\partial_x T_R, \partial_x T_R\rangle - \langle Q_2^{-1}\partial_y T_R, \partial_y T_R\rangle &\\ = 16((\mathcal{Q}(x, y) - R^2)^2\, \mathcal{Q}_1(x) - (\mathcal{Q}(x, y) + R^2)^2\, \mathcal{Q}_2(y)) &\\ = 16(\mathcal{Q}(x, y)^3 - 2R^2\,\mathcal{Q}(x, y)(\mathcal{Q}_1(x) + \mathcal{Q}_2(y)) + R^4\mathcal{Q}(x, y)) &\\ = 16\mathcal{Q}(x, y)\, T_R(x, y). &\qquad (4.5)\end{aligned}$$

This confirms that the regular zero set Σ_R of T_R is characteristic, and it is also clear that T_R has no critical point except the origin and $S_1 \cup S_2$.

EXAMPLE 4.1. For the Lorentz metric $L(x) = x_1^2 - x_2^2 - x_3^2 - x_4^2$ in $\mathbf{R}^4$ we can take for V_1 the $x_1 x_2$ plane and for V_2 the $x_3 x_4$ plane, which gives

$$T_R(x) = (x_1^2 - x_2^2 - x_3^2 - x_4^2)^2 - 2R^2(x_1^2 - x_2^2 + x_3^2 + x_4^2) + R^4.$$

The quadrics S_1 and S_2 become the hyperbola $\{(x_1, x_2, 0, 0);\, x_1^2 - x_2^2 = R^2\}$ and the circle $\{(0, 0, x_3, x_4);\, x_3^2 + x_4^2 = R^2\}$. The characteristic surface defined by $T_R(x) = 0$ occurs in the treatment by Riesz [R] of the progressive solutions of the wave equation in Friedlander [F]. The problem there was to determine simple layers v on Σ_R satisfying the wave equation. They are given by

$$v = \sqrt{\varrho(R + \varrho \sin\theta)}\, \exp(\pm i\theta/2 \pm i\beta/2)\, \delta(T_R),$$

if Σ_R is parametrized by

$$(\varrho, \theta, \beta) \mapsto (\varrho, \varrho\cos\theta, (R + \varrho\sin\theta)\cos\beta, (R + \varrho\sin\theta)\sin\beta)$$

with $\varrho \neq 0$ and $R + \varrho\sin\theta \neq 0$. These densities are smooth but two valued and we shall not discuss them further. However, the work of Friedlander and Riesz was the origin of this paper.

To study Σ_R near S_1 we shall write $x = (R + s)\,\alpha$ where $Q_1(\alpha) = 1$ and $s > -R$. Then

$$T_R(x, y) = T_R((R+s)\,\alpha, y) = (s^2 - \mathcal{Q}_2(y))((s + 2R)^2 - \mathcal{Q}_2(y)), \quad (4.6)$$

and the second factor is positive if $\mathcal{Q}_2(y) < R^2$ and $s > -R$. This shows that the regular zero set Σ_R of T_R is locally connected at S_1 unless $s^2 - \mathcal{Q}_2(y)$ has Lorentz signature or the opposite one, that is, unless $\mathcal{Q}_2$ is positive definite or of Lorentz signature. If $\mathcal{Q}_2$ is positive definite then $s = \sqrt{\mathcal{Q}_2(y)} \neq 0$ (resp. $s = -\sqrt{\mathcal{Q}_2(y)} \neq 0$) in the two components, which means that the second (resp. first) case in (4.4) occurs there. As remarked above, when $\mathcal{Q}_2$ is positive definite then the second case of (4.4) cannot overlap with the others outside S_1, so the second condition (4.4) is satisfied at every point in a component of Σ_R if it is satisfied at some point there. If $\mathcal{Q}_2$ has Lorentz signature we also have two components defined by $y \in \Lambda_2^{\pm}$ where $\Lambda_2^{\pm}$ are the two components of $\{y \in V_2;\, y \neq 0, \mathcal{Q}_2(y) \geqslant 0\}$. Note that when $T_R(x, y) = 0$ then y is in one of these cones or $y = 0$, thus $(x, y) \in S_1$, so y belongs to a fixed cone $\Lambda_2^{\pm}$ when (x, y) belongs to a component of Σ_R. When $\mathcal{Q}_1$ has Lorentz signature we define $\Lambda_1^{\pm}$ in the same way.

The preceding remarks make it easy to determine the components of Σ_R and also those of $T_R^+ = \{(x, y); T_R(x, y) > 0\}$ and of $T_R^- = \{(x, y); T_R(x, y) < 0\}$. The set T_R^+ contains all (x, y) with $Q_1(x) < 0$ or $Q_2(y) < 0$. In any component we can find a path close to the boundary up to the vicinity of $S_1 \cup S_2$, so the connectivity there is decisive. Altogether one gets the following picture of the number of components of Σ_R and of $T_R^\pm$ for different signatures of $\mathcal{Q}_j$:

	Number of components of Σ_R, T_R^+, T_R^-		
	$\mathcal{Q}_1$ pos. def.	$\mathcal{Q}_1$ Lorentz	$\mathcal{Q}_1$ other
$\mathcal{Q}_2$ pos. def.	3, 3, 1	4, 3, 2	2, 2, 1
$\mathcal{Q}_2$ Lorentz	4, 3, 2	4, 1, 4	2, 1, 2
$\mathcal{Q}_2$ other	2, 2, 1	2, 1, 2	1, 1, 1

As in Section 2 we shall now study homogeneous functions of T_R, so we calculate $\mathcal{Q}^*(\partial)\, T_R$, that is,

$$\begin{aligned}
&\operatorname{Tr} Q_1^{-1} \partial_x^2 T_R - \operatorname{Tr} Q_2^{-1} \partial_x^2 T_R \\
&\quad = 4(\operatorname{Tr}((\mathcal{Q}(x, y) - R^2) \operatorname{Id}_{V_1}) + 2\langle Q_1 x, x\rangle \\
&\qquad + \operatorname{Tr}((\mathcal{Q}(x, y) + R^2) \operatorname{Id} V_2) - 2\langle Q_2 y, y\rangle) \\
&\quad = 4\mathcal{Q}(x, y)(\dim V + 2) + 4R^2(\dim V_2 - \dim V_1). \qquad (4.7)
\end{aligned}$$

If $\dim V_1 = \dim V_2$, which implies that $\dim V$ is even and can always be attained then, it follows that

$$\mathcal{Q}^*(\partial)\, f(T_R) = 4\mathcal{Q}(x, y)((\dim V + 2)\, f'(T_R) + 4T_R\, f''(T_R))$$

vanishes if $T_R\, f''(T_R) = -\frac{1}{4}(\dim V + 2)\, f'(T_R)$, that is, $f'(T_R)$ is homogeneous of degree $-(\dim V + 2)/4$. *If f is homogeneous of degree $(2 - \dim V)/4$ and* $\dim V_1 = \dim V_2$ *it follows that $f(T_R)$ is defined outside the quadrics* (4.1) *and is annihilated by* $\mathcal{Q}^*(\partial)$. Then we get solutions $\chi_\pm^{(2-n)/4}(T_R)$ supported by Σ_R if $n/2$ is odd and supported by $\overline{T_R^\pm} \backslash (S_1 \cup S_2)$ when $n/2$ is even. From now on *we assume that* $\dim V_1 = \dim V_2 = \nu = n/2$. By (4.6) where the second factor is equal to $(2R)^2$ on S_1, we can use Theorem 3.3 to extend these distributions to a neighborhood of S_1 and apply $\mathcal{Q}^*(\partial)$ to the extension. The hypotheses of Proposition 3.2 are satisfied in a neighborhood of S_1 by $F = T_R$, with ν replaced by $\nu + 1$. The signature of

F'' is equal to $(1+\nu_2^-, \nu_2^+)$ if the signature of $\mathcal{Q}_2$ is (ν_2^+, ν_2^-). From Proposition 3.2 we obtain using the map $(x, y) \mapsto (s, y) \in \mathbf{R}^{\nu+1}$ that

$$\delta_{S_1, T_R} = \delta(s)\, \delta_{0, \mathcal{Q}_2}(y)/(2R)^{\nu+1},$$

where $\delta_{0, \mathcal{Q}_2}(y) = \delta(y)/\sqrt{|\det Q_2|}$ if Q_2 is the symmetric matrix of $\mathcal{Q}_2$ in a coordinate system in V_2. Since $\mathcal{Q}^*(\partial)\, T_R = 8R^2(\nu+1)$ on S_1, we have $\gamma = 4R^2$ in (3.5), so (3.7) gives the part of (4.8) and (4.9) in Theorem 4.2 below which refers to S_1. The other part follows since T_R is symmetric in $\mathcal{Q}_1$ and $\mathcal{Q}_2$ but interchanging them changes the sign of $\mathcal{Q}^*(\partial)$. Thus we have:

THEOREM 4.2. *Assume that* $\dim V_1 = \dim V_2 = \nu$ *and let the signature of the nondegenerate quadratic form* $\mathcal{Q}_j$ *in* V_j *be* (ν_j^+, ν_j^-) *where* $\nu_j^+ \geqslant 1$. *If* $T_R(x, y)$ *is defined by* (4.3) *when* $(x, y) \in V_1 \oplus V_2 = V$ *and* $\mathcal{Q}(x, y) = \mathcal{Q}_1(x) - \mathcal{Q}_2(y)$ *in* V, *then the distributions* $u_\pm = \chi_\pm^{(1-\nu)/2}(\pi T_R/4R^2)$ *are defined in* V *outside the singular quadrics* S_j *in* (4.1) *and they satisfy the equation* $\mathcal{Q}^*(\partial)\, u_\pm = 0$ *there. By Theorem* 3.3 *there is a natural extension* $U_\pm$ *of* $u_\pm$ *to* V, *and we have*

$$\mathcal{Q}^*(\partial)\, U_+ = 4 \sin(\pi(1+\nu_2^-)/2)(2R\delta(\mathcal{Q}_1(x) - R^2)) \cdot \delta_{0, \mathcal{Q}_2}(y) - 4 \sin(\pi(1+\nu_1^-)/2)\, \delta_{0, \mathcal{Q}_1}(x) \cdot (2R\delta(\mathcal{Q}_2(y) - R^2)), \quad (4.8)$$

$$\mathcal{Q}^*(\partial)\, U_- = -4 \sin(\pi\nu_2^+/2)(2R\delta(\mathcal{Q}_1(x) - R^2)) \cdot \delta_{0, \mathcal{Q}_2}(y) + 4 \sin(\pi\nu_1^+/2)\, \delta_{0, \mathcal{Q}_1}(x) \cdot (2R\delta(\mathcal{Q}_2(y) - R^2)). \quad (4.9)$$

When ν *is odd then* $U_- = (-1)^{(\nu-3)/2}\, U_+$, *and* $\operatorname{supp} U_+ \subset \overline{\Sigma_R} = \{(x, y);\ T_R(x, y) = 0\}$. *There is a natural decomposition of* U_+ *into a sum of distributions each supported by the closure of a component of the regular zero set* $\Sigma_R \setminus (S_1 \cup S_2)$, *and for such a term the contribution in the right-hand side of* (4.8) *should be omitted at* (*a component of*) S_j *which is not in the support and be divided by* 2 *when it is in the support of two terms. When* ν *is even there is a natural decomposition of* $U_\pm$ *as a sum of terms each of which is supported by the closure of a component of* $T_R^\pm = \{(x, y);\ \pm T_R(x, y) > 0\}$. *For such a term the right-hand sides of* (4.8), (4.9) *should be modified as when* ν *is odd.*

The preceding theorem contains Theorem 2.1; the proof using Theorem 3.3 is of course closely related.

Remark 4.3. When ν is odd the distributions on $\mathbf{R}$ which are homogeneous of degree $(1-\nu)/2$ are spanned by $\chi_+^{(1-\nu)/2}$ and

$$\underline{x}^{(1-\nu)/2} = ((x+i0)^{(1-\nu)/2} + (x-i0)^{(1-\nu)/2})/2.$$

Using Theorem 3.3 we can extend $\underline{T_R^{(1-\nu)/2}}$ from $\mathbf{R}^{2\nu}\backslash(S_1 \cup S_2)$ to $u \in \mathscr{D}'(\mathbf{R}^{2\nu})$ and obtain

$$\begin{aligned}
&\mathscr{Q}^*(\partial)\underline{(\pi T_R/4R^2)^{(1-\nu)/2}} \\
&\quad = -4\pi/\Gamma((\nu-1)/2)(\cos(\pi\nu_2^+/2)(2R\delta(\mathscr{Q}_1(x)-R^2))\cdot\delta_{0,\mathscr{Q}_2}(y) \\
&\qquad -\cos(\pi\nu_1^+/2)\,\delta_{0,\mathscr{Q}_1}(x)\cdot(2R\delta(\mathscr{Q}_2(y)-R^2))).
\end{aligned}$$

We shall not repeat the details of the proof.

Riesz [R, p. 854] observed that a Lorentz inversion at a point in the complement maps the characteristic surface in Example 4.1 to another surface of the same kind. The reason for that is that light rays are mapped to light rays since the Lorentz form is linear on them. We shall now give an analogue for the general case studied here.

First recall that in V inversion with respect to the metric form $\mathscr{Q}$ at the origin is defined by

$$V \ni X \mapsto \tilde{X} = R^2 X/\mathscr{Q}(X) \in V, \tag{4.10}$$

where $R > 0$. It is not defined when $\mathscr{Q}(X) = 0$, but since $\mathscr{Q}(\tilde{X}) = R^4/\mathscr{Q}(X)$, it is an involution in $\{X;\, \mathscr{Q}(X) > 0\}$ and in $\{X;\, \mathscr{Q}(X) < 0\}$,

$$X = R^2\tilde{X}/\mathscr{Q}(\tilde{X}). \tag{4.11}$$

As in the Euclidean case the involution is $\mathscr{Q}$-conformal, for

$$d\tilde{X} = R^2(dX/\mathscr{Q}(X) - X\, d\mathscr{Q}(X)/\mathscr{Q}(X)^2) \quad \text{implies}$$

$$\mathscr{Q}(d\tilde{X}) = R^4(\mathscr{Q}(dX)/\mathscr{Q}(X)^2 - 2\langle QX, dX\rangle/\mathscr{Q}(X)^3\, d\mathscr{Q}(X) + (d\mathscr{Q}(X))^2/\mathscr{Q}(X)^3),$$

and since $d\mathscr{Q}(X) = 2\langle QX, dX\rangle$ it follows that

$$\mathscr{Q}(d\tilde{X}) = R^4\mathscr{Q}(X)^{-2}\,\mathscr{Q}(dX). \tag{4.12}$$

When $X = (x, y)$ and $\mathscr{Q}(X) = \mathscr{Q}_1(x) - \mathscr{Q}_2(y)$ as above, and T_R is defined by (4.3), we obtain

$$T_R(\tilde{X}) = (R^2/\mathscr{Q}(X))^2\, T_R(X), \tag{4.13}$$

so the zeros of T_R are invariant under inversion with respect to $\mathscr{Q}$ where it is defined. However, the components of the regular zeros are not. If $\mathscr{Q}_1(x) \geqslant 0$, $\mathscr{Q}_2(y) \geqslant 0$ and $\sqrt{\mathscr{Q}_1(x)} \geqslant R + \sqrt{\mathscr{Q}_2(y)}$, then $\mathscr{Q}(x, y) = \mathscr{Q}_1(x) - \mathscr{Q}_2(y) \geqslant R^2 + 2R\sqrt{\mathscr{Q}_2(y)} \geqslant R^2 > 0$. For $\tilde{X} = (\tilde{x}, \tilde{y})$ we have $\tilde{x} = R^2x/\mathscr{Q}(x, y)$, $\tilde{y} = R^2y/\mathscr{Q}(x, y)$,

$$\sqrt{\mathcal{Q}_1(\tilde{x})} = R^2\sqrt{\mathcal{Q}_1(x)}/\mathcal{Q}(x, y), \quad \sqrt{\mathcal{Q}_2(\tilde{y})} = R^2\sqrt{\mathcal{Q}_2(y)}/\mathcal{Q}(x, y), \quad \text{hence}$$

$$\sqrt{\mathcal{Q}_1(\tilde{x})} + \sqrt{\mathcal{Q}_2(\tilde{y})} = R^2(\sqrt{\mathcal{Q}_1(x)} + \sqrt{\mathcal{Q}_2(y)})/(\mathcal{Q}_1(x) - \mathcal{Q}_2(y))$$

$$= R^2/(\sqrt{\mathcal{Q}_1(x)} - \sqrt{\mathcal{Q}_2(y)}) \leqslant R, \text{ and } \mathcal{Q}(\tilde{x}, \tilde{y}) > 0.$$

The calculation can be reversed, so the second set in (4.4) is mapped to the part of the first set in (4.4) with $\mathcal{Q}(x, y) > 0$; similarly the third set is mapped to the part where $\mathcal{Q}(x, y) < 0$. It is clear that the components of the set where $T_R(x, y) \neq 0$ are mapped analogously.

To exploit (4.13) we must recall the Kelvin transformation which is familiar in the Euclidean case. With the general notation above it consists in composition of a function u in V with the map $X \mapsto \tilde{X}$ and multiplication by an appropriate weight factor,

$$\tilde{u}(X) = u(R^2X/\mathcal{Q}(X))(R^2/\mathcal{Q}(X))^{(n-2)/2}, \tag{4.14}$$

where $n = \dim V$. Since $R^2/\mathcal{Q}(X) = (R^2/\mathcal{Q}(\tilde{X}))^{-1}$, this is an involution, that is, $\tilde{\tilde{u}} = u$. It is of course only defined where $\mathcal{Q}(X) \neq 0$, and it depends on a choice of argument for $\mathcal{Q}(X)$ unless n is even. The important point is that

$$\mathcal{Q}^*(\partial)\,\tilde{u}(X) = (R^2/\mathcal{Q}(X))^2\,(\widetilde{\mathcal{Q}^*(\partial)\,u})(X), \qquad u \in C^2(V), \tag{4.15}$$

or written out in detail

$$\mathcal{Q}^*(\partial)(u(R^2X/\mathcal{Q}(X))\,\mathcal{Q}(X)^{(2-n)/2}) = R^4\mathcal{Q}(X)^{-(n+2)/2}\,(\mathcal{Q}^*(\partial)\,u)(R^2X/\mathcal{Q}(X)). \tag{4.15$'$}$$

In the proof we may assume that $\mathcal{Q}$ is diagonalized: $\mathcal{Q}(X) = \sum q_j X_j^2$, $\mathcal{Q}^*(\partial) = \sum q_j^{-1}\partial_j^2$. An easy computation gives (see [H1, p. 138])

$$\mathcal{Q}^*(\partial)\,\mathcal{Q}^a = (2n + 4(a-1))\,a\mathcal{Q}^{a-1} = \begin{cases} 0, & \text{if } a = (2-n)/2 \\ 2n\mathcal{Q}^{-(n+2)/2}, & \text{if } a = -n/2. \end{cases}$$

Hence $\mathcal{Q}^*(\partial)\,\mathcal{Q}(X)^{(2-n)/2} = 0$ (as for the familiar Newton potential), and differentiation gives that $\mathcal{Q}^*(\partial)(X_j\mathcal{Q}(X)^{-n/2}) = 0$ for every j, which proves (4.15)$'$ when u is a first order polynomial. Another differentiation gives

$$\delta_{jk}\mathcal{Q}^*(\partial)\,\mathcal{Q}(X)^{-n/2} - n\mathcal{Q}^*(\partial)(\mathcal{Q}(X)^{-(n+2)/2}\,q_k X_j X_k) = 0.$$

If $u(X)=\sum a_{jk}X_jX_k$ it follows that

$$n\mathcal{Q}^*(\partial)(\mathcal{Q}(X)^{-(n+2)/2}\,u(X))=\left(\sum a_{jj}/q_j\right)\mathcal{Q}^*(\partial)\,\mathcal{Q}(X)^{-n/2}$$
$$=n(\mathcal{Q}^*(\partial)\,u)\,\mathcal{Q}(X)^{-(n+2)/2},$$

which proves that (4.15)′ is valid when u is a quadratic form. Hence (4.15)′ follows for a general $u\in C^2$, for it is valid for the second order Taylor polynomial at $\tilde{X}$. An extension of (4.15) to distributions u follows at once by continuity. For the distribution $u=\frac{1}{2}\chi_+^{(2-n)/4}(\pi T_R/4R^2)$ defined outside $S_1\cup S_2$ we obtain using (4.13)

$$\tilde{u}=\tfrac{1}{2}\chi_+^{(2-n)/4}(\pi T_R(R^2/\mathcal{Q})^2/4R^2)(R^2/\mathcal{Q})^{(n-2)/2}$$
$$=\tfrac{1}{2}\chi_+^{(2-n)/4}(\pi T_R/4R^2)=u,$$

so u is invariant under the Kelvin transformation.

Inversion at a point X^0 is of course defined by making a preliminary translation of X^0 to the origin before applying the map (4.10), and perhaps making another translation afterwards. We shall now determine the inversion of the zeros of T_R, defined by (4.3), at a point (x^0, y^0) with $T_R(x^0, y^0)\neq 0$. To simplify notation somewhat we define the inversion now by (4.10) with R replaced by 1. Thus $T_R(x+x^0, y+y^0)$ is changed to

$$\mathcal{Q}(x, y)^2\,T_R(x/\mathcal{Q}(x, y)+x^0,\ y/\mathcal{Q}(x, y)+y^0), \tag{4.16}$$

and we have:

Proposition 4.4. *If $T_R(x^0, y^0)\neq 0$ then the polynomial* (4.16) *is equal to*

$$T_R(x^0, y^0)\,\mathcal{Q}(x-x^1, y-y^1)^2-2R^2\mathcal{A}(x-x^1, y-y^1)+R^4/T_R(x^0, y^0), \tag{4.17}$$

where

$$x^1=x^0(R^2-\mathcal{Q}(x^0, y^0))/T_R(x^0, y^0),\quad y^1=-y^0(R^2+\mathcal{Q}(x^0, y^0))/T_R(x^0, y^0), \tag{4.18}$$

and $\mathcal{A}$ is the quadratic form

$$\mathcal{A}(x, y)=\mathcal{Q}_1(x)+\mathcal{Q}_2(y)+8(\langle Q_1x, x^0\rangle^2\,\mathcal{Q}_2(y^0)+\mathcal{Q}_1(x^0)\langle Q_2y, y^0\rangle^2$$
$$+\langle Q_1x, x^0\rangle\langle Q_2y, y^0\rangle(R^2-\mathcal{Q}_1(x^0)-\mathcal{Q}_2(y^0)))/T_R(x^0, y^0), \tag{4.19}$$

which is the sum of the quadratic forms

$$\widetilde{\mathscr{Q}}_1(x, y)=\tfrac{1}{2}(\mathscr{A}(x, y)+\mathscr{Q}(x, y)), \qquad \widetilde{\mathscr{Q}}_2(x, y)=\tfrac{1}{2}(\mathscr{A}(x, y)-\mathscr{Q}(x, y))$$

while their difference is equal to $\mathscr{Q}$. If $\tilde{V}_j$ is the radical of $\widetilde{\mathscr{Q}}_{3-j}$, $j=1, 2$, then $V=\tilde{V}_1\oplus\tilde{V}_2$ is a $\mathscr{Q}$ orthogonal decomposition, and $\widetilde{\mathscr{Q}}_j$ is nondegenerate in $\tilde{V}_j$. If $T_R(x^0, y^0)>0$ then the signature of $\tilde{Q}_j$ in $\tilde{V}_j$ is equal to that of $\mathscr{Q}_j$ in V_j, and the inverted surface is then after a translation and a $\mathscr{Q}$ orthogonal transformation the zero surface of T_r where $r=R/\sqrt{T_R(x^0, y^0)}$. If $T_R(x^0, y^0)<0$ then $\mathscr{Q}_1(x^0)>0$ and $\mathscr{Q}_2(y^0)>0$, the transformed surface is still of the same form with $r=R/\sqrt{|T_R(x^0, y^0)|}$, and $\widetilde{\mathscr{Q}}_1$, $\widetilde{\mathscr{Q}}_2$ replaced by $-\widetilde{\mathscr{Q}}_2$, $-\widetilde{\mathscr{Q}}_1$, with signatures (ν_2^-+1, ν_2^+-1), (ν_1^-+1, ν_1^+-1) if (ν_j^+, ν_j^-) is the signature of $\mathscr{Q}_j$. They are equal to those of $\mathscr{Q}_1$, $\mathscr{Q}_2$ if and only if $\nu_2^-+1=\nu_1^+$, $\nu_1^-+1=\nu_2^+$, which means that the signatures ν^+, ν^- of $\mathscr{Q}$ are odd and that

$$\nu_1^+=(1+\nu^+)/2, \qquad \nu_1^-=(\nu^--1)/2, \qquad \nu_2^+=(1+\nu^-)/2, \qquad \nu_2^-=(\nu^+-1)/2,$$

thus $\dim V_1=\dim V_2$. *In this case the transformed surface is always equal to T_r with $r=R/\sqrt{|T(x^0, y^0)|}$ after a translation and a $\mathscr{Q}$ orthogonal transformation.*

Proof. Postponing the proof of (4.17), (4.18), (4.19) we note that if $\mathscr{Q}_1(x^0)\neq 0$ then

$$\begin{aligned}\widetilde{\mathscr{Q}}_1(x, y)=(\mathscr{Q}_1(x)-\langle Q_1x, x^0\rangle^2/\mathscr{Q}_1(x^0))+\mathscr{Q}_1(x^0)(2\langle Q_2y, y^0\rangle\\ +\langle Q_1x, x^0\rangle(R^2-\mathscr{Q}_1(x^0)-\mathscr{Q}_2(y^0))/\mathscr{Q}_1(x^0))^2/T_R(x^0, y^0),\end{aligned}$$

for $4\mathscr{Q}_1(x^0)\,\mathscr{Q}_2(y^0)-(\mathscr{Q}_1(x^0)+\mathscr{Q}_2(y^0)-R^2)^2=-T_R(x^0, y^0)$. The first term is equal to $\mathscr{Q}_1$ in the $\mathscr{Q}_1$ orthogonal space of x^0 in V_1, and x^0 is in its radical. Hence the rank of $\widetilde{\mathscr{Q}}_1$ is at most equal to that of $\mathscr{Q}_1$. Since the rank of $\widetilde{\mathscr{Q}}_2$ is at most equal to that of $\mathscr{Q}_2$, and since $\widetilde{\mathscr{Q}}_1-\widetilde{\mathscr{Q}}_2=\mathscr{Q}$, it follows that the ranks are equal, and so are the signatures if $T_R(x^0, y^0)>0$. If $T_R(x^0, y^0)<0$, thus $\mathscr{Q}_1(x^0)>0$ then the positive signature of $\widetilde{\mathscr{Q}}_1$ is decreased by one and the negative one is increased by 1. Division by the negative quantity $T_R(x^0, y^0)$ in (4.17) requires a change of sign and an interchange of $\widetilde{\mathscr{Q}}_1$, $\widetilde{\mathscr{Q}}_2$. The signatures are unchanged if $\nu_2^-+1=\nu_1^+$ and $\nu_1^-+1=\nu_2^+$, which gives $\nu^+=\nu_1^++\nu_2^-=2\nu_1^+-1$ and $\nu^-=\nu_1^-+\nu_2^+=2\nu_1^-+1$. Thus ν^+ and ν^- are odd and $\dim V=\nu^++\nu^-=2\dim V_1$, so $\dim V_1=\dim V_2$, and we obtain $\nu_2^+=(1+\nu^-)/2$ and $\nu_2^-=(\nu^+-1)/2$ as claimed.

For reasons of continuity the preceding results remain true for arbitrary (x^0, y^0) with $T_R(x^0, y^0) \neq 0$. The rest of the statement is obvious so it just remains to prove (4.17), (4.18), (4.19). It suffices to do so when $\mathcal{Q}_1(x^0) > 0$ and $\mathcal{Q}_2(y^0) > 0$. We can then choose coordinates x in V_1 and y in V_2 such that $x^0 = (a, 0, ..., 0)$, $y^0 = (b, 0, ..., 0)$, and $\mathcal{Q}_1(x) = x_1^2 + q_1$, $\mathcal{Q}_2(y) = y_1^2 + q_2$ where q_1 and q_2 are quadratic forms in the other variables x' and y'. Then the polynomial (4.16) becomes

$$\begin{aligned}
&((a^2 - b^2)\, \mathcal{Q}(x, y) + 2(ax_1 - by_1) + 1)^2 \\
&\qquad - 2R^2((a^2 + b^2)\, \mathcal{Q}(x, y)^2 + 2(ax_1 + by_1)\, \mathcal{Q}(x, y) + x_1^2 + x_2^2 + q_1 + q_2) \\
&\qquad + R^4 \mathcal{Q}(x, y)^2 \\
&\quad = T_R(x^0, y^0)(\mathcal{Q}(x, y) + ((1 + 2(ax_1 - by_1))(a^2 - b^2) \\
&\qquad - 2R^2(ax_1 + by_1))/T_R(x^0, y^0))^2 \\
&\qquad - ((1 + 2(ax_1 - by_1))(a^2 - b^2) - 2R^2(ax_1 + by_1))^2/T_R(x^0, y^0) \\
&\qquad + (1 + 2(ax_1 - by_1))^2 - 2R^2(x_1^2 + y_1^2 + q_1 + q_2).
\end{aligned}$$

The first term on the right is equal to the first term in (4.17) since

$$a^2(R^2 - a^2 + b^2)^2 - b^2(a^2 - b^2 + R^2)^2 = (a^2 - b^2)\, T_R(x^0, y^0).$$

What remains is just to verify (4.17) in the case where $V_1 = V_2 = \mathbf{R}$ and $\mathcal{Q}_1(x) = x^2$, $\mathcal{Q}_2(y) = y^2$ which is easily done using a computer. However, it is more illuminating to do so by clarifying the geometrical contents. We can write (cf. (2.4))

$$\begin{aligned}
T_R(x, y) &= (x^2 - y^2 - R^2 + 2Ry)(x^2 - y^2 - R^2 - 2Ry) \\
&= (x - y + R)(x + y - R)(x - y - R)(x + y + R),
\end{aligned}$$

so the zeros are the characteristic lines forming a parallelogram with center at the origin and vertices $(0, \pm R)$ and $(\pm R, 0)$. Moving the origin to (a, b) changes them to $(-a, \pm R - b)$ and $(-a \pm R, -b)$. Inversion maps the characteristic lines to characteristic lines forming a parallelogram with opposite vertices

$$\begin{aligned}
&(-a, \pm R - b)/(a^2 - b^2 - R^2 \pm 2Rb) \\
&\quad = (-a, \pm R - b)(a^2 - b^2 - R^2 \mp 2Rb)/T_R(a, b) \\
&\quad = (a(R^2 - a^2 + b^2), -b(a^2 - b^2 + R^2))/T_R(a, b) \\
&\qquad \pm R(2ba, a^2 + b^2 - R^2)/T_R(a, b).
\end{aligned}$$

The center of the parallelogram is the point in (4.18), and moving the origin there the equation of the inverted zero set becomes, with $T = T_R(a, b)$,

$$\begin{aligned}
&((x - 2Rba/T)^2 - (y - R(a^2 + b^2 - R^2)/T)^2) \\
&\qquad \times ((x + 2Rba/T)^2 - (y + R(a^2 + b^2 - R^2)/T)^2) \\
&\quad = (x^2 + 4R^2a^2b^2/T^2 - y^2 - R^2(a^2 + b^2 - R^2)^2/T^2)^2 \\
&\qquad - 4R^2(2bax - (a^2 + b^2 - R^2)\, y)^2/T^2 \\
&\quad = (x^2 - y^2 - R^2/T)^2 - 4R^2(2bax - (a^2 + b^2 - R^2)\, y)^2/T^2 = 0.
\end{aligned}$$

After multiplication by $T = T_R(a, b)$ we get the equation

$$\begin{aligned}
&T_R(a, b)(x^2 - y^2)^2 + R^4/T_R(a, b) \\
&\qquad = 2R^2(x^2 - y^2 + 2(2bax - (a^2 + b^2 - R^2)\, y)^2/T_R(a, b)).
\end{aligned}$$

Apart from the factor $2R^2$ the coefficient of x^2 in the right-hand side is $1 + 8a^2b^2/T_R(a, b)$, that of xy is $-8ab(a^2 + b^2 - R^2)/T_R(a, b)$, and that of y^2 is

$$\begin{aligned}
&-1 + 2(a^2 + b^2 - R^2)^2/T_R(a, b) \\
&\qquad = 1 + 2((a^2 + b^2 - R^2)^2 - T_R(a, b))/T_R(a, b) = 1 + 8a^2b^2/T_R(a, b),
\end{aligned}$$

which completes the proof of (4.17).

Next we shall calculate the result of making an inversion with center at a point (x^0, y^0) where $T_R = 0$ but $T'_R \neq 0$. Then we have $\mathcal{Q}_1(x^0) \geqslant 0$ and $\mathcal{Q}_2(y^0) \geqslant 0$, and at first we assume that both inequalities are strict. We can then choose the coordinates as in the proof of (4.19), now with $a + b = R$, and (4.16) becomes

$$\begin{aligned}
&((a^2 - b^2)\, \mathcal{Q}(x, y) + 2(ax_1 - by_1) + 1)^2 \\
&\qquad - 2R^2((a^2 + b^2)\, \mathcal{Q}(x, y)^2 + 2(ax_1 + by_1)\, \mathcal{Q}(x, y) + x_1^2 + y_1^2 + q_1 + q_2) \\
&\qquad + R^4 \mathcal{Q}(x, y)^2 \\
&\quad = 2R(a - b - 4ab(x_1 + y_1))\, \mathcal{Q}(x, y) + (2(ax_1 - by_1) + 1)^2 \\
&\qquad - 2R^2(x_1^2 + y_1^2 + q_1 + q_2),
\end{aligned}$$

where we have used that

$$2(a^2 - b^2)(2(ax_1 - by_1) + 1) - 4R^2(ax_1 + by_1) = 2R(a - b) - 8abR(x_1 + y_1)$$

since $a+b=R$. If we set $x_1+y_1=\sigma+(a-b)/4ab$ and $x_1-y_1=\tau$, then the right-hand side becomes

$$-8abR\sigma(q_1-q_2)-2R^2(q_1+q_2)-8abR\sigma(\sigma+(a-b)/4ab)\,\tau \\ +(\sigma(a-b)+\tau R+R^2/4ab)^2-R^2((\sigma+(a-b)/4ab)^2+\tau^2).$$

The last three terms can be simplified to $-8abR(\sigma^2-R^2/16a^2b^2)(\tau+1/2R)$. With the notation $\gamma=R/4ab$ and replacing $\tau+1/(2R)$ by τ, we obtain the equation

$$\tau(\sigma^2-\gamma^2)+(\gamma+\sigma)\,q_1(x')+(\gamma-\sigma)\,q_2(y')=0,$$

after dividing by $-8abR$. Returning to the coordinates $x_1=(\sigma+\tau)/2$ and $y_1=(\sigma-\tau)/2$, which means a translation of the original coordinates, we obtain the equation

$$(x_1-y_1)((x_1+y_1)^2-\gamma^2)+(\gamma+x_1+y_1)\,q_1(x')+(\gamma-x_1-y_1)\,q_2(y')=0. \tag{4.20}$$

The singular quadrics have become the paraboloids

$$\begin{aligned} &x_1+y_1=\gamma, && x'=0, && 2\gamma(x_1-y_1)-q_2(y')=0, \quad \text{and} \\ &x_1+y_1=-\gamma, && y'=0, && 2\gamma(x_1-y_1)-q_1(x')=0. \end{aligned} \tag{4.21}$$

If $\mathcal{Q}_2$ is indefinite then $X^0=(x^0,y^0)$ is also a regular zero of T_R if $\mathcal{Q}_1(x^0)=R^2$ and $\mathcal{Q}_2(y^0)=0$, $y^0\neq 0$. (There is a similar case with $\mathcal{Q}_1$ and $\mathcal{Q}_2$ interchanged). As before we can choose the coordinates so that $x^0=(R,0,...,0)$ and $\mathcal{Q}_1(x)=x_1^2+q_1(x')$, but now we choose the y coordinates so that y^0/R is the unit vector on the y_2 axis and $Q_2(y)=2y_1y_2+q_2(y'')$ where $y''=(y_3,...,y_{n_2})$. Then (4.16) becomes

$$\begin{aligned}(R^2\mathcal{Q}(x,y)&+2R(x_1-y_1)+1)^2-2R^2(R^2\mathcal{Q}(x,y)^2+2R(x_1+y_1)\,\mathcal{Q}(x,y)\\ &+x_1^2+2y_1y_2+q_1+q_2)\;R^4\mathcal{Q}(x,y)^2\\ &=2R^2(1-4Ry_1)\,\mathcal{Q}(x,y)+(2R(x_1-y_1)+1)^2\\ &\quad-2R^2(x_1^2+2y_1y_2+q_1+q_2).\end{aligned}$$

If we set $y_1=\tilde{y}_1+1/4R$, $y_2=\tilde{y}_2-1/4R$, $x_1=\tilde{x}_1-1/2R$ and divide by $-8R^3$, the equation of the inverted surface becomes

$$\begin{aligned}(\gamma+y_1)\,q_1(x')&+(\gamma-y_1)(q_2(y'')-x_1^2)-2y_2(y_1^2-\gamma^2)\\ &=y_1\mathcal{Q}(x,y)+2y_2\gamma^2-\gamma(x_1^2-q_2(y'')-q_1(x'))=0,\end{aligned} \tag{4.22}$$

where $\gamma = 1/4R$. This is of the same form as (4.20); the singular paraboloids are

$$\begin{aligned} &y_1 = \gamma, \qquad x' = 0, \qquad x_1^2 - q_2(y'') - 4\gamma y_2 = 0, \\ &y_1 = -\gamma, \qquad x_1 = 0, \qquad y'' = 0, \qquad q_1(x') + 4\gamma y_2 = 0. \end{aligned} \tag{4.23}$$

In Example 4.1 with the Lorentz metric in $\mathbf{R}^4$, (4.20) and (4.22) are in the parabolic case of Friedlander [F] and Riesz [R].

Finally we shall determine the inversion at a point in $S_1 \cup S_2$, for example $(x^0, 0)$ where $Q_1(x^0) = R^2$. We choose the x coordinates as above so that $x^0 = (R, 0, ..., 0)$ and $\mathcal{Q}_1(x) = x_1^2 + q_1(x')$. Then (4.16) becomes

$$\begin{aligned} &(R^2\mathcal{Q}(x, y) + 2Rx_1 + 1)^2 - 2R^2(R^2\mathcal{Q}(x, y)^2 + 2Rx_1\mathcal{Q}(x, y) \\ &\quad + \mathcal{Q}_1(x) + \mathcal{Q}_2(y)) + R^4\mathcal{Q}(x, y)^2 = -4R^2\mathcal{Q}_2(y) + (2Rx_1 + 1)^2. \end{aligned}$$

Hence the inverted surface is defined by

$$\hat{T}(x, y) = (x_1 + 1/2R)^2 - \mathcal{Q}_2(y) = 0. \tag{4.24}$$

For the Lorentz metric in $\mathbf{R}^4$ this was called the cylindrical case in [R].

The surfaces (4.20) and (4.22) obtained by inversion of the zero set of T_R at a regular point suggest another way of splitting the vector space V which leads to a cubic polynomial instead of T_R. Let $\mathcal{Q}$ still be a non-singular indefinite quadratic form in a vector space V of finite dimension n but choose now a two dimensional subspace V_0 where $\mathcal{Q}$ is indefinite and extend to a $\mathcal{Q}$ orthogonal decomposition

$$V = V_0 \oplus V_1 \oplus V_2, \qquad \text{thus} \quad \mathcal{Q} = \mathcal{Q}_0 + \mathcal{Q}_1 + \mathcal{Q}_2$$

where $\mathcal{Q}_j$ is a non-singular quadratic form in V_j. Choose coordinates $t = (t_1, t_2)$ in V_0 such that $\mathcal{Q}_0(t) = 2t_1 t_2$ and set with $x \in V_1$, $y \in V_2$, and a parameter $a \neq 0$,

$$\begin{aligned} P(t, x, y) &= t_1\mathcal{Q}(x, y) + a(\mathcal{Q}_1(x) - \mathcal{Q}_2(y)) - 2a^2 t_2 \\ &= 2(t_1^2 - a^2)\, t_2 + (t_1 + a)\, \mathcal{Q}_1(x) + (t_1 - a)\, \mathcal{Q}_2(y). \end{aligned} \tag{4.25}$$

The equation (4.20) is of the form $P = 0$ if $x_1 + y_1 = t_1$, $x_1 - y_1 = 2t_2$, with q_1, $-q_2$ in the roles of $\mathcal{Q}_1$, $\mathcal{Q}_2$ in (4.25). The equation (4.22) is of this form if we change the sign of y_1 and let $x_1^2 - q_2(y'')$, $q_1(x')$ play the roles of $\mathcal{Q}_1$, $\mathcal{Q}_2$. (Such surfaces were also encountered in [H2]. The parameter a is actually irrelevant and disappears if t_1 is replaced by at_1 and t_2 is replaced by t_2/a.) With the notation used in Proposition 4.4 we can interpret

(4.20) and (4.22) as follows. If $T_R(x^0, y^0)=0$ then $x^0 \neq 0$ and $y^0 \neq 0$ if (x^0, y^0) is not on a singular quadric, and

$$T'_R(x^0, y^0)=4((\mathcal{Q}_1(x^0)-\mathcal{Q}_2(y^0)-R^2)\,Q_1 x^0, (\mathcal{Q}_2(y^0)-\mathcal{Q}_1(x^0)-R^2)\,Q_2 y^0).$$

If the first component vanishes then $\mathcal{Q}_1(x^0)-\mathcal{Q}_2(y^0)=R^2$ which implies $\mathcal{Q}_2(y^0)=0$ and $\mathcal{Q}_1(x^0)=R^2$. Similarly $\mathcal{Q}_1(x^0)=0$ and $\mathcal{Q}_2(y^0)=R^2$ if the second component vanishes. If both are different from 0 then $\mathcal{Q}_1(x^0)>0$, $\mathcal{Q}_2(y^0)>0$; the annihilator of $Q_1 x^0$ in V_1, that is, the subspace $\tilde{V}_1$ of V_1 which is $\mathcal{Q}_1$ orthogonal to x^0, the subspace $\tilde{V}_2$ of V_2 which is $\mathcal{Q}_2$ orthogonal to y^0, and the space $\tilde{V}_0$ spanned by x^0 and y^0 have the roles in (4.25). If instead the first component of T'_R vanishes then we can choose $y^1 \in V_2$ not $\mathcal{Q}_2$ orthogonal to y^0, and let $\tilde{V}_0$ be the space spanned by y^0 and y^1, let $\tilde{V}_1$ be the subspace of V_1 which is $\mathcal{Q}_1$ orthogonal to x^0, and let $\tilde{V}_2$ be the space spanned by x^0 and the subspace of V_2 which is $\mathcal{Q}_2$ orthogonal to y^0 and y^1. Note that the signatures of the forms obtained in the two constructions are the same, so all the inverted surfaces are equivalent.

The equation $\partial P/\partial t_2=0$ implies $t_1=\pm a$, so the equations $\partial P/\partial t_1=4t_1 t_2 + \mathcal{Q}_1(x)+\mathcal{Q}_2(y)=0$ show that the critical points of P are the paraboloids

$$\begin{aligned} S_1&=\{(a, t_2, 0, y); 4at_2+\mathcal{Q}_2(y)=0\},\\ S_2&=\{(-a, t_2, x, 0); 4at_2-\mathcal{Q}_1(x)=0\}, \end{aligned} \tag{4.26}$$

and $P=0$ in S_1 and in S_2.

With P defined by (4.25) we have

$$\begin{aligned} &2\partial P/\partial t_1 \partial P/\partial t_2+\mathcal{Q}_1^*(\partial P/\partial x)+\mathcal{Q}_2^*(\partial P/\partial y)\\ &\quad=2(4t_1 t_2+\mathcal{Q}_1(x)+\mathcal{Q}_2(y))\cdot 2(t_1^2-a^2)\\ &\qquad+4((t_1+a)^2\,\mathcal{Q}_1(x)+(t_1-a)^2\,\mathcal{Q}_2(y))=8t_1 P(t, x, y), \end{aligned} \tag{4.27}$$

$$\begin{aligned} &2\partial^2 P/\partial t_1 \partial t_2+\mathcal{Q}_1^*(\partial_x)\,P+\mathcal{Q}_2^*(\partial_y)\,P\\ &\quad=8t_1+2(t_1+a)\dim V_1+2(t_1-a)\dim V_2\\ &\quad=2(n+2)\,t_1+2a(\dim V_1-\dim V_2). \end{aligned} \tag{4.28}$$

When $\dim V_1=\dim V_2=\nu$, thus $n=2+2\nu$, it follows that

$$\mathcal{Q}^*(\partial)\,f(P)=2t_1(n+2)\,f'(P)+8t_1 P f''(P)=0$$

if f' is homogeneous of degree $-(n+2)/4$, hence if f is homogeneous of degree $(2-n)/4$. Thus $u_\pm=\chi_\pm^{-\nu/2}(P)$ is defined outside $S_1 \cup S_2$ and is there annihilated by $\mathscr{Q}^*(\partial)=2\partial^2/\partial t_1\partial t_2+\mathscr{Q}_1^*(\partial/\partial x)+\mathscr{Q}_2^*(\partial/\partial y)$. Introducing new coordinates $s=(s_1, s_2)$ by

$$t_1=a+s_1, \qquad t_2=s_2-\mathscr{Q}_2(y)/(2s_1+4a) \tag{4.29}$$

in a neighborhood of S_1, but keeping the coordinates x and y we obtain

$$P=(2a+s_1)(2s_1s_2+\mathscr{Q}_1(x)), \tag{4.30}$$

and S_1 is defined by $s=0$ and $x=0$. The hypotheses of Proposition 3.2, with ν replaced by $\nu+2$, are satisfied by $F=P$ in view of (4.30), and $\delta_{S,P}=\delta(s_1)\,\delta(s_2)\,\delta_{0,\mathscr{Q}_1}(x)(2a)^{-(\nu+2)/2}$. Since $\mathscr{Q}^*(\partial)\,P=2(n+2)\,a=4(\nu+2)\,a$ we have $\gamma=2a$ in (3.5). Thus Theorem 3.3 defines $\chi_{\pm}^{-\nu/2}(P)$ as a distribution also near S_1 and gives the first term in

$$\begin{aligned}\mathscr{Q}^*(\partial)\,\chi_{\pm}^{-\nu/2}(\pi P/2a)&\\ =\pm 16a(\sin(\pi(1+\nu_1^{\pm})/2)\,&\delta(t_1-a)\cdot\delta(4at_2+\mathscr{Q}_2(y))\cdot\delta_{0,\mathscr{Q}_1}(x)\\ -\sin(\pi(1+\nu_2^{\mp})/2)\,&\delta(t_1+a)\cdot\delta(4at_2-\mathscr{Q}_1(x))\cdot\delta_{0,\mathscr{Q}_2}(y)).\end{aligned} \tag{4.31}$$

If we replace t by $-t$ and P by $-P$, then the roles of $\mathscr{Q}_1$ and $\mathscr{Q}_2$ are interchanged so the second term in (4.31) is justified. We have proved:

THEOREM 4.5. *Assume that* $\dim V_1=\dim V_2=\nu$, *that* $V_0=\mathbf{R}^2$ *and that* $\mathscr{Q}(t,x,y)=2t_1t_2+\mathscr{Q}_1(x)+\mathscr{Q}_2(y)$ *with* $t\in V_0$, $x\in V_1$ *and* $y\in V_2$. *Denote the signature of the nondegenerate quadratic form* $\mathscr{Q}_j$ *in* V_j *by* (ν_j^+, ν_j^-), *and let* P *be defined by* (4.25). *Then the distributions* $u_\pm=\chi_{\pm}^{-\nu/2}(\pi P/2a)$ *defined outside the singular paraboloids* (4.26) *have a natural extension to distributions* $U_\pm$ *in the whole space, and* (4.31) *is then valid for the extensions. When the support with* S_j *removed is not connected at* S_j *there is a natural decomposition into a sum of two terms. For each of them* (4.31) *is fulfilled with the factor* 16 *replaced by* 8.

Writing the equation $P(t,x,y)=0$ in the form

$$2t_2+\mathscr{Q}_1(x)/(t_1-a)+\mathscr{Q}_2(y)/(t_1+a)=0$$

suggests a generalisation where we decompose V into a direct sum of a number of $\mathscr{Q}$ orthogonal spaces

$$V=V_0\oplus V_1\oplus\cdots\oplus V_\mu$$

with V_0 as before and the variable in V_j denoted by x_j. Thus $\mathcal{Q} = \mathcal{Q}_0(t) + \mathcal{Q}_1(x_1) + \cdots + \mathcal{Q}_\mu(x_\mu)$ where $\mathcal{Q}_0(t) = 2t_1 t_2$. Let $a_1 < a_2 < \cdots < a_\mu$ and set

$$\psi(t_1, x_1, ..., x_\mu) = \tfrac{1}{2} \sum_1^\mu \mathcal{Q}_j(x_j)/(t_1 - a_j).$$

Then the surface where $t_2 + \psi(t_1, x_1, ..., x_\mu) = 0$ is characteristic with respect to $\mathcal{Q}^*(\partial)$ when t_1 is not a zero of $A(t_1) = \prod_1^\mu (t_1 - a_j)$, for

$$2\partial\psi/\partial t_1 = -\sum_1^\mu \mathcal{Q}_j(x_j)/(t_1 - a_j)^2, \qquad \mathcal{Q}_j^*(\partial\psi/\partial x_j) = \mathcal{Q}_j(x_j)/(t_1 - a_j)^2.$$

For the polynomial

$$P = 2A(t_1)(t_2 + \psi(t_1, x_1, ..., x_\mu)) = 2A(t_1)\, t_2 + \sum_1^\mu \mathcal{Q}_j(x_j) \prod_{i;\, i \neq j} (t_1 - a_i) \tag{4.32}$$

we obtain

$$\begin{aligned} &2\partial P/\partial t_1 \partial P/\partial t_2 + \sum_1^\mu \mathcal{Q}_j^*(\partial P/\partial x_j) \\ &\quad = 8A(t_1)(A(t_1)\, \partial\psi/\partial t_1 + (t_2 + \psi)\, A'(t_1)) + 4A(t_1)^2 \sum_1^\mu \mathcal{Q}_j^*(\partial\psi/\partial x_j) \\ &\quad = 4A'(t_1)\, P. \end{aligned} \tag{4.27$'$}$$

The critical points of P are the paraboloids

$$t_1 = a_i,\ x_i = 0,\ 2t_2 + \sum_{j;\, j \neq i} \mathcal{Q}_j(x_j)/(a_i - a_j) = 0, \tag{4.33}$$

where $i = 1, ..., \mu$, for $\partial P/\partial t_2 = 0$ implies $A(t_1) = 0$, hence $t_1 = a_i$ for some i. The equation $\partial P/\partial x_i = 0$ gives $x_i = 0$, and the equation $\partial P/\partial t_1 = 0$ then gives the remaining equation. We have $P = 0$ on all the paraboloids, but all other zeros of P are regular. (Note that the affine subspaces defined by $t_1 = a_i$ and $x_i = 0$ are zeros of P.)

With $v_j = \dim V_j$, hence $n = 2 + \sum_1^\mu v_j$, we have

$$\mathcal{Q}^*(\partial)\, P = 2A(t_1) \sum_1^\mu v_j/(t_1 - a_j) + 4A'(t_1).$$

This is equal to $2(\nu_j+2)\,A'(t_1)$ when $t_1=a_j$, so it is a constant times $A'(t_1)$ if and only if $\nu_1=\cdots=\nu_\mu=\nu$ which implies that $n-2=\nu\mu$ so μ has to be a factor of $n-2$. In that case $\mathcal{Q}^*(\partial)\,P=2(\nu+2)\,A'(t_1)$ and

$$\mathcal{Q}^*(\partial)\,f(P)=2(\nu+2)\,A'(t_1)\,f'(P)+4A'(t_1)\,Pf''(P)=0$$

if f' is homogeneous of degree $-\nu/2-1$, hence if f is homogeneous of degree $-\nu/2$. This means that $\chi_\pm^{-\nu/2}$ are distribution solutions defined outside the singular paraboloids. As in the proof of Theorem 4.5 we can extend to distributions in V by making the following change of coordinates when t_1 is close to a_i:

$$\begin{aligned} t_1&=a_i+s_1,\\ t_2&=s_2-\tfrac{1}{2}\sum_{j\neq i}\mathcal{Q}_j(x_j)/(t_1-a_j)\\ &=s_2-\tfrac{1}{2}\sum_{j\neq i}\mathcal{Q}_j(x_j)/(s_1+a_i-a_j), \end{aligned}$$

for then we obtain

$$P=(A(a_i+s_1)/s_1)(2s_1s_2+\mathcal{Q}_i(x_i)),$$

where $A(a_i+s_1)/s_1=A'(a_i)$ when $s_1=0$. We can again apply Theorem 3.3, with ν replaced by $\nu+2$, taking the sign $\sigma_i=\pm$ of $A'(a_i)$ into account. This gives an extension of $\chi_\pm^{-\nu/2}(P)$ to a neighborhood of the singular paraboloid where $t_1=a_i$ and gives an extension of (4.31):

$$\begin{aligned} &\mathcal{Q}^*(\partial)\,\chi_\pm^{-\nu/2}(\pi P)\\ &\quad=\sum_{i=1}^{\mu}C_i\delta(t_1-a_i)\,\delta\Big(t_2+\tfrac{1}{2}\sum_{j\neq i}\mathcal{Q}_j(x_j)/(a_i-a_j)\Big)\,\delta_{0,\mathcal{Q}_i}(x_i),\\ &\qquad C_i=\pm 4\sigma_i|A'(a_i)|^{-\nu/2}\sin(\pi(1+\nu_i^{\pm\sigma_i})/2). \end{aligned}$$

If we make an inversion of the surface $P=0$ at a point outside the surface we obtain a further generalisation of the surfaces in the first part of this section. However, we shall not pursue this idea for it is quite clear that for higher degrees and dimensions there is a rich supply of polynomials which lead to solutions of second order homogeneous differential equations like those studied in this section. We shall also refrain from discussing surfaces such as (4.24) for this leads only to well known fundamental solutions of $\mathcal{Q}^*(\partial)$ in fewer variables.

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